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ABSTRACT

Solutions were obtained of the wave equation for a plane horizontally polarized electro-magnetic wave incident on a semi infinite two dimensional inhomogeneous medium. The complex dielectric constant of the medium was assumed to be $\epsilon(z) = \epsilon'(0)(1 + az) - i\,\epsilon''(0)(1 + bz)$, where a, and b are positive constants. Two problems were considered: an inhomogeneous half space, and an inhomogeneous layer of arbitrary thickness for $0 \leq z \leq z_1$, contiguous to an inhomogeneous half space for $z_1 \leq z < \infty$. The dielectric constant $\epsilon(z)$ is discontinuous at $z = 0$ but continuous at $z = z_1$, the gradient, grad $\epsilon(z)$, is discontinuous at $z = 0, z_1$. Solutions of the wave equation were obtained in terms of Hankel functions with complex arguments. Numerical calculations were made of the reflection coefficient R at the interface of the homogeneous medium, $z = 0$, as a function of a, b, $\epsilon'(0)$, $\epsilon''(0)$, by programming the Hankel functions and the expressions for R on a 360/91 computer. The startling results are that the reflection coefficient for a complex dielectric constant with gradient, $\epsilon'(0)\,a - i\,\epsilon''(0)\,b$, can be less than that of the same medium with zero gradient, i.e. $\epsilon(z) = \epsilon'(0) - i\,\epsilon''(0)$. The physical explanation of this behavior is given in terms of the interference of the coherent scattering reflections which take place thruout the inhomogeneous medium.

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CONTENTS

REFLECTION AND INTERFERENCE OF ELECTROMAGNETIC WAVES IN INHOMOGENEOUS MEDIA

I. INTRODUCTION

The problem of a plane electromagnetic wave incident on a semi infinite inhomogeneous medium has a long and interesting history. The problem is of two kinds, the purely analytical one of solving the wave equation with a variable coefficient describing the inhomogeneity of the medium and the physical interpretation of the results.

Exact solutions have been obtained in a small number of cases, and various approximations which place a limit on the variation of the dielectric constant over the wavelength. The physical interpretation of the results would appear to be fairly clear, early doubts as to the presence of reflections in the inhomogeneous medium were dispelled by Wallot [1], but primarily by Bremmer [2]. Similarly "reflections" at discontinuities of gradients, and the question of unique resolution of the wave field into incident and reflected waves have been clarified[3]. But the full implications of the effect of internal reflections in the inhomogeneous medium seem not to have been appreciated.

Microwave measurements of reflectivities of soils, and soils with varying amounts of moisture as a function of depth, from satellite and aircraft platforms made it necessary to get both insight into the physical nature of the problem, and actual figures of reflection coefficients. We therefore investigated the solution of the wave equation for semi infinite and layered media with linearly increasing dielectric constant in some detail. The solution of the problem for a real dielectric constant was obtained by Gans[4], and Wallot [1].

Wallot obtained a closed solution in terms of Hankel functions for an inhomogeneous transition layer between two homogeneous media. The real dielectric constant is assumed to increase continuously from the free space value ϵ_0 to the value ϵ_1 of the second homogeneous medium. As the transition layer is reduced to zero thickness the Fresnel reflection coefficient is found to increase monotonically from zero to the limiting value corresponding to the Fresnel coefficient between two homogeneous media of dielectric constants ϵ_0 and ϵ_1, respectively. If we suppose that the transition layer is replaced by a single semi infinite inhomogeneous layer, Wallot's results clearly indicate that the reflection coefficient at the interface increases monotonically to the limiting value of 1 as the gradient approaches infinity.

In our formulation of the problem we shall use the mathematical formalism of Brekhovskikh[5] as the most suitable for our purposes.

Brekhovskikh calculates the reflection coefficient for a layered, and semi infinite inhomogeneous medium. Like Wallot, he uses a real dielectric constant and continuity of $\epsilon(z)$ at the interface between homogeneous and inhomogeneous layer. In our calculation, we will change the physical characteristics of the inhomogeneous layer in three important respects, a) the dielectric constant is discontinuous at the first interface between homogeneous and inhomogeneous layer or inhomogeneous semi infinite medium, b) the dielectric constant is complex, c) the real and imaginary parts of the dielectric constant vary linearly with z, but independently of each other. As shown above, Wallots results indicate then that the reflection coefficient for an inhomogeneous half space with a complex dielectric constant will only differ from those with a real constant in detail but not in essentials. In other words we expect the absolute reflection coefficient to increase monotonically to the limiting value 1 as grad $\epsilon(z)$ increases without limit.

In fact the reflection coefficient behaves in a quite unexpected fashion depending on the individual gradients of $\epsilon'(z)$ and $\epsilon''(z)$. The reflection coefficient is no longer a monotonically increasing function of $\epsilon'(z)$ and $\epsilon''(z)$*. We will show that the behavior of the reflection coefficient can be explained by interference phenomena in the inhomogeneous medium, and that Wallot's conclusions apply only to a special case.

Early investigators were handicapped in the interpretation of their results by the unwieldiness of their results for the reflection coefficient. Hankel functions had only been tabulated for integral and half integral orders. Consequently only a small number of calculations were made under special limiting conditions, i.e. very large or very small arguments of the Hankel functions. We have made machine calculations of Hankel functions for complex arguments of order 1/3 and 2/3, and programmed our reflection coefficient calculations on an IBM 360/91 computer.

II. THE WAVE EQUATION AND WAVE FIELDS FOR AN INHOMOGENEOUS MEDIUM

Consider a plane wave incident from free space on a semi infinite inhomogeneous medium whose dielectric properties are a function of z only. (See Figure 1.) The wave is polarized in the y-direction, the electric fields in the

*We were not able to prove analytically the monotonic increase of the absolute value of the Fresnel reflection coefficient with increasing ϵ' and ϵ'' of the dielectric constant $\epsilon' - i\epsilon''$, but computer runs of $|R|$ as a function of ϵ' and ϵ'' tend to show this.

x- and z-directions are zero, i.e. the incident field is horizontally polarized. The inhomogeneous halfspace may be separated by an inhomogeneous transition layer between $0 \leq z \leq z_1$ from free space. Both inhomogeneous media have a dielectric constant which is a linear complex function of z. Designating free space, the inhomogeneous layer, and the inhomogeneous halfspace as media I, II, and III, respectively, $\epsilon(z)$ for medium II will be,*

$$\epsilon(z) = \epsilon'(z) - i\epsilon''(z) = \epsilon'(0)(1 + az) - i\epsilon''(0)(1 + bz)$$

$$0 \leq z \leq z_1, \quad a \geq 0, \quad b \geq 0,$$

and in medium III,

$$\epsilon(z) = \epsilon'(0)\left((1 + az_1) + c(z - z_1)\right) - i\epsilon''(0)\left((1 + bz_1) + d(z - z_1)\right),$$

$$z_1 \leq z < \infty, \quad c \geq 0, \quad d \geq 0.$$

It will be seen that $\epsilon(z)$ is a continuous function, but the gradient, $\nabla\epsilon(z)$, is discontinuous at $z = z_1$.

The electric field E_y satisfies the general wave equation[5]

$$\partial^2 E_y/\partial z^2 + E_y\left(k^2(z) - \sin^2\theta_0\right) = 0 \tag{1}$$

where,

$$k^2(z) = \omega^2\mu_0\epsilon(z) \qquad 0 \leq z < \infty, \quad \text{media II and III,} \tag{2}$$

and, θ_0, angle of incidence of plane wave.

*MKS units and a positive time factor $\exp(i\omega t)$ are used thruout. See References 6 and 7.

In medium I, $k^2(z)$ reduces to the free space wave number $\omega^2 \mu_0 \epsilon_0$, where,

$$\omega = 2\pi\nu,$$

the circular frequency, sec^{-1},

$$\mu_0 = 1.257 \times 10^{-6}\ Hm^{-1},$$

the permeability of free space,

$$\epsilon_0 = 8.854 \times 10^{-12}\ Fm^{-1},$$

the dielectric constant of free space.

The solution of Eq. (1) in medium I is given by the incident and reflected wave-field,

$$E_y = \exp(-i k_0 \cos\theta_0 z) + R \exp(i k_0 \cos\theta_0 z), \tag{3}$$

where, R is the reflection coefficient at $z = 0$. In medium II, we have the wave equation,

$$\partial^2 E_y/\partial z^2 + k_0^2 \left[\epsilon'(0)/(1 + a z)/\epsilon_0 - i\,\epsilon''(0)(1 + b z)/\epsilon_0 - \sin^2\theta_0\right] E_y = 0 \tag{4}$$

where,

$$k_0^2 = \omega^2 \mu_0 \epsilon_0,$$

the free space wave number

and,

$$k^2(z) = \omega^2 \mu_0 \left(\epsilon'(0)\,(1 + a z) - i\,\epsilon''(0)\,(1 + b z)\right), \tag{5}$$

$$\epsilon(0) = \epsilon'(0) - i\,\epsilon''(0),$$

the dielectric constant of the inhomogeneous medium at the interface $z = 0$.

We introduce the variable,

$$\xi = \epsilon'(0)\,(1 + a z)/\epsilon_0 - i\,\epsilon''(0)\,(1 + b z)/\epsilon_0 - \sin^2\theta_0, \tag{6}$$

and rewrite Eq. (4),

$$\partial^2 E_y/\partial \xi^2 + \left(k_0/\left(a\,\epsilon'(0)/\epsilon_0 - i\,b\,\epsilon''(0)/\epsilon_0\right)\right)^2 \xi\, E_y = 0. \tag{7}$$

We make a second transformation,

$$w = 2/3 \left(\frac{k_0}{a\,\epsilon'(0)/\epsilon_0 - i\,b\,\epsilon''(0)/\epsilon_0}\right) \xi^{3/2}, \tag{8}$$

writing the wave equation in terms of the new variable w,

$$\partial^2 E_y/\partial w^2 + 1/3\; w^{-1}\,\partial E_y/\partial w + E_y = 0. \tag{9}$$

The solution of Eq. (9) can be written most conveniently for our purpose in terms of Hankel functions[8] of the first and second kind of order 1/3,

$$E_y = A\, w^{1/3} H_{1/3}^{(1)}(w) + B\, w^{1/3} H_{1/3}^{(2)}(w),^{9} \qquad 0 \le z \le z_1. \tag{10}$$

Similarly we have for medium III,

$$\partial^2 E_y/\partial z^2 + k_0^2 \left[\epsilon'(0)\ \big((1 + a z_1) + c\,(z - z_1)\big)/\epsilon_0 \right.$$
$$\left. - i\,\epsilon''(0)\ \big((1 + b z_1) + d\,(z - z_1)\big)/\epsilon_0 - \sin^2\theta_0\right] E_y = 0. \tag{11}$$

Making the transformations as above,

$$\eta = \epsilon'(0)\ \big((1 + a z_1) + c\,(z - z_1)\big)/\epsilon_0$$
$$- i\,\epsilon''(0)\ \big((1 + b z_1) + d\,(z - z_1)\big)/\epsilon_0 - \sin^2\theta_0, \tag{12}$$

$$v = 2/3\left(\frac{k_0}{c\,\epsilon'(0)/\epsilon_0 - i\,d\,\epsilon'(0)/\epsilon_0}\right)\eta^{3/2}, \tag{13}$$

$$E_y = C\,v^{1/3}\,H_{1/3}^{(1)}(v) + D\,v^{1/3}\,H_{1/3}^{(2)}(v), \tag{14}$$

where C and D are constants as are A and B in Eq. (10).

IIa. Determination of the Constants A, B, C, D and the Reflection Coefficient R

The electric field E_y must remain finite as z goes to infinity. From Eqs. (12) and (13) we have,

$$\lim_{z \to \infty} v = 2/3\ k_0\ z^{3/2}\ \big(\epsilon'(0)\,c/\epsilon_0 - i\,\epsilon''(0)\,d/\epsilon_0\big),$$

or $v \to \infty$ as $z \to \infty$, and v will be in the fourth quadrant,

$$(-1/2\,\pi \le \arg v < 0)\ \text{for}\ c > 0, \quad d > 0, \quad \epsilon'/\epsilon_0 \ge 1, \quad \epsilon''/\epsilon_0 \ge 0;$$

hence $H_{1/3}^{(1)}(v) \to \infty$, and $H_{1/3}^{(2)} \to 0$ as $v \to \infty$, and we must have[10] $C = 0$. And we have,

$$E_y = D\, v^{1/3} H_{1/3}^{(2)}(v), \qquad z_1 \leq z < \infty. \tag{15}$$

The continuity conditions for E_y and $\partial E_y/\partial z \,\alpha\, H_x$ at $z = z_1$ give the following equations,

$$A\, w^{1/3} H_{1/3}^{(1)}(w) + B\, w^{1/3} H_{1/3}^{(2)}(w) = D\, v^{1/3} H_{1/3}^{(2)}(v), \qquad z = z_1, \tag{16}$$

$$z_1 = z\Big|_{z=z_1}, \qquad v_1 = v\Big|_{z=z_1}, \qquad w_1 = w\Big|_{z=z_1},$$

$$\frac{dw}{dz}\left(A\, w^{1/3} H_{-2/3}^{(1)}(w) + B\, w^{1/3} H_{-2/3}^{(2)}(w)\right) = \frac{dv}{dz} D\, v^{1/3} H_{-2/3}^{(2)}(v), \tag{17}$$

$$z = z_1,$$

where we have used the relation,[11]

$$\frac{d}{dw}\left(w^{1/3} H_{1/3}^{(1),(2)}(w)\right) = w^{1/3} H_{-2/3}^{(1),(2)}(w). \tag{18}$$

From Eqs. (6), (8), (12), and (13),

$$\frac{dw}{d\xi}\,\frac{d\xi}{dz} = k_0 \xi^{1/2}, \qquad \frac{dv}{d\eta}\,\frac{d\eta}{dz} = k_0 \eta^{1/2}, \tag{19}$$

and

$$\frac{dw}{dz} = \frac{dv}{dz}, \qquad z = z_1.$$

Solving Eqs. (16), and (17) for D,

$$D = \left(A w_1^{1/3} H_{1/3}^{(1)}(w_1) + B w_1^{1/3} H_{1/3}^{(2)}(w_1)\right) / \left(v_1^{1/3} H_{1/3}^{(2)}(v_1)\right), \tag{20}$$

we find for A/B from Eq. (16),

$$A/B = \frac{H_{1/3}^{(2)}(w_1) H_{-2/3}^{(2)}(v_1) - H_{-2/3}^{(2)}(w_1) H_{1/3}^{(2)}(v_1)}{H_{-2/3}^{(1)}(w_1) H_{1/3}^{(2)}(v_1) - H_{-2/3}^{(2)}(v_1) H_{1/3}^{(1)}(w_1)} . \tag{21}$$

The continuity conditions at the interface z = 0 give from Eqs. (3) and (10),

$$(1 + R) = A w^{1/3} H_{1/3}^{(1)}(w) + B w^{1/3} H_{1/3}^{(2)}(w) \Big|_{z=0} \tag{22}$$

and

$$w_0 = w \Big|_{z=0} ,$$

and

$$i k_0 (R - 1) \cos \theta_0 = \frac{dw}{dz} \left(A w^{1/3} H_{-2/3}^{(1)}(w) + B w^{1/3} H_{-2/3}^{(2)}(w)\right) \Bigg|_{\substack{z=0 \\ w=w_0}} , \tag{23}$$

$$\frac{dw}{dz}\Bigg|_{z=0} = k_0 \left(\epsilon'(0)/\epsilon_0 - i \epsilon''(0)/\epsilon_0 - \sin^2 \theta_0\right)^{1/2}. \tag{24}$$

From Eqs. (23) and (24) we easily find for the reflection coefficient at z = 0,

$$R = \frac{- i \cos \theta_0 - \left(\epsilon'(0)/\epsilon_0 - i \epsilon''(0)/\epsilon_0 - \sin^2 \theta_0\right)^{1/2} \cdot G}{- i \cos \theta_0 + \left(\epsilon'(0)/\epsilon_0 - i \epsilon''(0)/\epsilon_0 - \sin^2 \theta_0\right)^{1/2} \cdot G} , \tag{25}$$

where,

$$G = \frac{(A/B)\, H_{-2/3}^{(1)}(w_0) + H_{-2/3}^{(2)}(w_0)}{(A/B)\, H_{1/3}^{(1)}(w_0) + H_{1/3}^{(2)}(w_0)} . \tag{26}$$

Equation (25) looks very much like the Fresnel equation for reflection of a plane wave from a semi infinite medium with dielectric constant $\epsilon'(0) - i\,\epsilon''(0)$, with a factor $G\,(w_0, w_1, v_1)$ in the numerator and denominator to account for the inhomogeneity. In fact we can show that Eq. (25) reduces to the Fresnel equation if the arguments of the Hankel functions are allowed to go to infinity, i.e. $a = b = c = d \to 0$, then $|w_0| \to \infty$, $|w_1| \to \infty$, and $|v_1| \to \infty$. In other words the inhomogeneity has been removed.

Using the asymptotic expansion of Hankel functions[12] for large arguments, we find,

$$H_{1/3}^{(1)}(z) = (2/\pi z)^{1/2} \exp(i z - i\pi 5/12) \left(1 - i \cdot 5/(72 z)\right), \tag{27}$$

$$H_{-2/3}^{(1)}(z) = (2/\pi z)^{1/2} \exp(i z + i\pi/12) \left(1 + i \cdot 7/(72 z)\right), \tag{28}$$

$$H_{1/3}^{(2)}(z) = (2/\pi z)^{1/2} \exp(- i z + i\pi 5/12) \left(1 + i \cdot 5/(72 z)\right), \tag{29}$$

$$H_{-2/3}^{(2)}(z) = (2/\pi z)^{1/2} \exp(- i z - i\pi/12) \left(1 - i \cdot 7\,(72 z)\right), \tag{30}$$

where all arguments of z are assumed to be in the 4th quadrant. After a great deal of algebraic manipulation we have,

$$A/B = (1/12) \exp(- i\, 2 w_1 + i\pi/3) \left(1/w_1 - 1/v_1 + 0\,(w_1^2, v_1^2)\right) \tag{31}$$

and

$$G = \frac{- i \left(1 - i/(6 w_0)\right) + (1/12) \exp\left(2 i (w_0 - w_1)\right) (1/w_1 - 1/v_1)}{1 - (i/12) \exp\left(2 i (w_0 - w_1)\right) (1/w_1 - 1/v_1)} . \tag{32}$$

In the limit as $|w_0|, |w_1|, |v_1| \to \infty$,

$$\lim_{\substack{w_0 \\ w_1 \\ v_1}\Big\}\to\infty} G = -i ,$$

and Eq. (25) reduces to the Fresnel equation,

$$R = \frac{\cos\theta_0 - \left(\epsilon'(0)/\epsilon_0 - i\,\epsilon''(0)/\epsilon_0 - \sin^2\theta_0\right)^{1/2}}{\cos\theta_0 + \left(\epsilon'(0)/\epsilon_0 - i\,\epsilon''(0)/\epsilon_0 - \sin^2\theta_0\right)^{1/2}} . \tag{33}$$

IIb. Limiting Expressions for the Reflection Coefficient

The double layer expression for the reflection coefficient R, Eq. (25), can be reduced to a single inhomogeneous layer (i.e. inhomogeneous half space) expression by setting $c = a$, and $d = b$, then w_0 remains unchanged, and $w_1 = v_1$. Equation (21) reduces to zero and G becomes,

$$G = H_{-2/3}^{(2)}(w_0)/H_{1/3}^{(2)}(w_0) . \tag{34}$$

If the first layer, $(0 \le z \le z_1)$, is inhomogeneous, and the half space, $(z_1 \le z < \infty)$, homogeneous, $c = d = 0$, $v_1 \to \infty$, and we have the following expressions for A/B and G,

$$A/B = \frac{-\,i\,H_{1/3}^{(2)}(w_1) - H_{-2/3}^{(2)}(w_1)}{H_{-2/3}^{(1)}(w_1) + i\,H_{1/3}^{(2)}(w_1)} \tag{35}$$

$$G = \frac{(A/B)\,H_{-2/3}^{(1)}(w_0) - H_{-2/3}^{(2)}(w_0)}{(A/B)\,H_{1/3}^{(1)}(w_0) - H_{1/3}^{(2)}(w_0)} . \tag{36}$$

No useful limiting expression is found if the first layer is homogeneous and the second an inhomogeneous halfspace. In this case $a = b = 0$, w_0 and w_1 approa infinity. This problem has to be solved by considering the specific boundary value problem from the beginning, and solving the equations for the reflection coefficient.

IIc. Machine Calculation of the Reflection Coefficient

The expression for the reflection for an inhomogeneous medium is so complex that no conclusions can be drawn about the general behavior of the coefficient. Equations (25), (26), and (21) were therefore programmed for an IBM 360 computer. The program included also direct calculation of the Hankel functions of the first and second kind of orders 1/3 and 2/3. Tabulations of these functions are not available except for modified Hankel functions with complex arguments and absolute values $|z| \leq 6$. This is much too restrictive a range for the problem under consideration, and for a general exploration of the behavior of R. We therefore wrote our own straightforward computer program using the following equations to calculate the Hankel functions for an arbitrary complex argument.[13] For $|z| < 7.5$ we used,

$$H_\nu^{(1)}(z) = i \csc(\nu\pi) \left[\exp(-i\pi\nu)\, J_\nu(z) - J_{-\nu}(z)\right] \qquad (37$$

$$H_\nu^{(2)}(z) = -i \csc(\nu\pi) \left[\exp(i\pi\nu)\, J_\nu(z) - J_{-\nu}(z)\right], \qquad (38$$

where the Bessel functions were calculated using the ascending series,

$$J_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{m=0}^{\infty} \frac{(-1/4\, z^2)^m}{m!\,\Gamma(\nu + m + 1)} . \qquad (39$$

The series was summed in double precision arithmetic to yield J (z) to 6 1/2 significant figures. For $|z| \geq 7.5$ we used terms through μ^4 in the asymptotic expansion of the Hankel functions:

$$H_\nu^{(1)} = \sqrt{2/(\pi z)} \left[P(\nu, z) + i\, Q(\nu, z)\right] \exp(i\chi) \qquad (40$$

$$H_\nu^{(2)}(z) = \sqrt{2/(\pi z)}\left[P(\nu, z) - i\,Q(\nu, z)\right] \exp(-i\chi), \qquad (41)$$

where,

$$\chi = z - (1/2\nu + 1/4)\pi, \qquad \mu = 4\nu^2,$$

$$P(\nu, z) = \sum_{k=0}^{\infty} (-1)^k \frac{(\nu, 2k)}{(2z)^{2k}},$$

$$Q(\nu, z) = \sum_{k=0}^{\infty} (-1)^k \frac{(\nu, 2k+1)}{(2z)^{2k+1}},$$

and,

$$(\nu, k) \equiv \frac{(\mu - 1^2)(\mu - 3^2)\ \ldots\ \left(\mu - (2k-1)^2\right)}{2^{2k}\,k!},$$

$$(\nu, 0) \equiv 1.$$

This prescription yields Hankel functions accurate to six significant figures for $|z| < 7.5$ or $|z| > 10$, and five significant figures for $7.5 \leq |z| \leq 10$. Greater accuracy can be obtained by retaining more terms in the expansions.

III. RESULTS OF COMPUTER CALCULATION OF THE REFLECTION COEFFICIENT

Machine calculations were made of the reflection coefficient for single and double layer inhomogeneous media at several microwave frequencies. The calculations on the single layer semi infinite inhomogeneous medium lend themselves more readily to interpretation and understanding of the underlying physical processes, and will be discussed first.

Figures 2, 3, and 4 show a plot of the reflection coefficient for an inhomogeneous half space and horizontal polarization. The dielectric constant at the vacuum-dielectric interface was chosen to correspond to soil with roughly 2% moisture, and alternately with 15% moisture by weight. Moisture content was assumed to increase as a function of depth (distance from interface) resulting in a hypothetical linear increase of $\epsilon'(z)$ and $\epsilon''(z)$, or constant gradient $\partial(z)/\partial z = \epsilon'(0)\,a - i\,\epsilon''(0)\,b$. The absolute amplitude reflection coefficient was plotted as a function of the constant b, which controls the gradient of the imaginary part of the dielectric constant, for various values of the constant a.

Intuitively one expects $|R|$ to increase from the value determined by the Fresnel equation for $\epsilon = \epsilon'(0) - i\,\epsilon''(0)$ with increasing a and/or b. Although there is an immediate increase in $|R|$ for $a = b > 0$, further increase in b results in a steady decrease of $|R|$ until a minimum is reached. We find the surprising fact that for a broad range of values of a and b the reflection coefficient at the interface drops considerably below the value of the Fresnel coefficient of a semi infinite homogeneous medium with dielectric constant $(\epsilon'(0) - i\,\epsilon''(0))$, i.e. the dielectric constant of the inhomogeneous at the vacuum-dielectric interface. On the other hand if $b = 0$, but not necessarily $\epsilon''(z)$ the reflection coefficient increases steadily with increasing a from the aforementioned value of the reflection coefficient. This behaviour is shown in Figure 5, and is entirely in accord with one's physical intuition.

One is forced to conclude, since the absolute value of the Fresnel coefficient is a monotonically increasing function of $\epsilon'(z)$ and $\epsilon''(z)$*, that the drop in the reflection coefficient of the inhomogeneous medium is caused by interference. Bremmer[2] explored the physical interpretation of the wave field in inhomogeneous media, and was able to demonstrate the presence of reflected waves at all points of the inhomogeneous medium. We will try to demonstrate in a subsequent section that the phase relationship of the reflections in the inhomogeneous medium is such as to cause interferences and a reduction in the amplitude of the wave emerging from the inhomogeneous halfspace.

A limited number of calculations were also made for a two-layered inhomogenous medium at three microwave frequencies, 37.0, 19.35, and 5.0 Gc/sec. The first inhomogeneous layer is of finite thickness and is followed by a semi infinite inhomogeneous medium. The dielectric constant is continuous at the interface between the two inhomogeneous media, but there is a discontinuity in the gradient of the dielectric constant, i.e. $\partial\epsilon/\partial z = \epsilon'(0)\,a - i\,\epsilon''(0)\,b$ for $0 \leq z \leq z_1$, and $\partial\epsilon/\partial z = \epsilon'(0)\,c - i\,\epsilon''(0)\,d$ for $z_1 \leq z < \infty$, where z_1 is the distance from the free space-dielectric interface to the interface between the inhomogeneous media.

Figure 6 shows the results obtained for a two layer model for two values of z_1. However the dielectric constant gradients and the dielectric constant $\epsilon(0) = \epsilon'(0) - i\,\epsilon''(0)$ at $z = 0$ were the same in both cases. For comparison we calculated the reflection coefficient as a function of frequency for a simple semi infinite inhomogeneous medium whose dielectric constant $\epsilon(z)$ is the same as

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*See footnote on p. 3.

that of the first layer of the above composite (two layer) model. There is a striking difference in the behavior of the reflection coefficient for the three cases. The single layer reflection coefficient is relatively insensitive to changes in frequency, the double layer, on the other hand, shows relatively large fluctuations in the coefficient when z_1 is changed from 0.5×10^{-2} to 0.25×10^{-2} meter*. We conclude, therefore, that the discontinuous change in the gradient $\partial \epsilon / \partial z$ at $z = z_1$ gives rise to strong reflections, which emerge at $z = 0$ and interferes with the reflection at the air-dielectric interface. A change in z_1 of 0.25 cm corresponds very roughly to a quarter wave length for 19 Gc/s and an average dielectric constant of $\epsilon_{ave} = 3.0 \times 10^{-11}$. Thus the emergent reflection will have suffered an approximate phase shift of π, and will be in phase with the reflection at the interface. This reasoning assumes that a definite reflected wave can be assumed at the interface $z = z_1$. However the wavefield in the first inhomogeneous layer cannot be separated into incident and reflected waves so that a reflection coefficient at $z = z_1$ has no meaning and cannot be defined.

IV. INTERFERENCE EFFECTS IN INHOMOGENEOUS MEDIA

The approximate solution of the wave equation (see Eq. (1)) for an inhomogeneous medium extending from $-\infty$ to $+\infty$ with a slowly varying wave number $k(z)$ is the well known WKB solution[14],

$$E_y = \left(k(z)\right)^{-1/2} \left[C_1 \exp\left(\int_0^z - i k(z)\, dz \right) + C_2 \exp\left(\int_0^z - i k(z)\, dz \right) \right], \tag{42}$$

where C_1 and C_2 are constants. There are no reflections to this order of approximation, and the two waves in the positive and negative directions, respectively, are independent of each other.[15] Bremmer[16] has shown that Eq. (42) represents only the zero order term in a series solution, and higher order approximations can be interpreted as reflected waves. Thus the zero order wave

$$\left(k(z)\right)^{-1/2} C_1^{(0)} \exp\left(- i \int_0^z k(z)\, dz \right),$$

propagating in the positive z-direction from $-\infty$, produces a reflection at each

*Suggested by T. Wilheit.

point in the medium given by

$$C_1^{(1)} = \int_{+\infty}^{z} C_1^{(0)} f(z) \exp\left(2i \int_0^z k(z)\, dz\right).$$

Conversely, the zero order wave

$$C_2^{(0)} \; k(z)^{-1/2} \exp\left(i \int_0^z k(z)\, dz\right)$$

produces a first order reflected wave propagating in the positive z-direction, etc. It can easily be shown that the zero order waves propagate without reflection. The power flow calculated from the Poyting vector

$$\vec{S} = (\vec{E}_y \times \vec{H}_x)$$

for the "upgoing" wave

$$(k(z))^{-1/2}\left(\exp \int_0^z - i k(z)\, dz\right)$$

is a constant, i.e. undiminished by reflections. In order to demonstrate the presence of interference effects in a semi infinite medium, as apparently manifested by the drop in the reflection coefficient at the vacuum-dielectric interface, one would have to show simply the presence of emerging scattering reflections such as $C_1^{(1)}$, $C_1^{(2)}$, ... etc. These obviously coherent reflections are then capable of interference. Their exact phase relationship will be determined solely by the dielectric constant $\epsilon(z)$ of the semi infinite inhomogeneous medium.

However, for a boundary value problem such as ours it is no longer possible to demonstrate in a perfectly general way the reflected waves in the inhomogeneou half space. The WKB solution of the wave equation for a plane wave incident on a semi infinite inhomogeneous space is,

$$E_y = C_1 \left(k(z)\right)^{-1/2} \exp\left(-i\int_0^z k(z)\,dz\right), \qquad z \geq 0.$$

Using Bremmer's approach in the preceding paragraph for an infinite medium ($-\infty < z < +\infty$) and applying it to the semi infinite case ($0 \leq z < \infty$) turns out to be not very fruitful. It seems impossible to arrive at higher order approximations to E_y, which could be identified as secondary, ... etc. waves originating from reflection losses of a primary, ... etc. wave. A more direct approach suggested by Bremmer's exact solution of a plane wave reflection from a semi infinite medium[17] with $k(z) \alpha\ 1/z$ consists in examining the reflection coefficient (see Eqs. (25) and (34)) directly.

Using the asymptotic expansion of $H_{1/3}^{(2)}$ and $H_{-2/3}^{(2)}$ for $|w| >> 1$ of Eqs. (40), and (41) one finds for G, (see Eq. (34)),

$$G = H_{-2/3}^{(2)}(w_0)/H_{1/3}^{(1)}(w_0) \sim \exp(-i\pi/2)\ \frac{(1 - i\,a_1/w_0 + a_2/w_0^2 \ldots)}{(1 + i\,b_1/w_0 - b_2/w_0^2 \ldots)}\ .$$

Using the binomial expansion, we have,

$$G \sim \exp(-i\pi/2)\ \left(1 - (a_1 + b_1)/w_0 + (a_2 + b_2 - a_1 b_1 - b_1^2)/w_0^2 + 0\,(w_0^3)\right)$$

where,

$$a_1 = 7/72$$

$$b_1 = 5/72$$

$$a_2 = 455/10368$$

$$b_2 = 385/10368\,.$$

From Eq. (25) we have for the reflection coefficient, for normal incidence,

$$R = \frac{1 - \left(\epsilon_r(0)\right)^{1/2} \left(1 - i\,(a_1 + b_1)/w_0 + (a_2 + b_2 - a_1 b_1 - b_1^2)/w_0^2 \ldots\right)}{1 + \left(\epsilon_r(0)\right)^{1/2} \left(1 - i\,(a_1 \quad b_1)/w_0 + \text{etc.} \ldots\right)} ,$$

where

$$\left(\epsilon_r(0)\right)^{1/2} \equiv \left(\epsilon'(0)/\epsilon_0 - i\,\epsilon''(0)/\epsilon_0\right)^{1/2}.$$

We can rewrite R by again using the binomial expansion and assuming

$$\left|\left(\epsilon_r(0)\right)^{1/2} / \left(1 + \epsilon_r(0)\right)^{1/2} (a_1 + b_1)/w_0\right| ,$$

and

$$\left|\left(\epsilon_r(0)\right)^{1/2} / \left(1 + \epsilon_r(0)\right)^{1/2} (a_2 + b_2 - a_1 b_1 - b_1^2)/w_0^2 \right| \ll 1.$$

Retaining only second order terms in w_0, and after much algebraic manipulation, we find,

$$R = \left(1 - \left(\epsilon_r(0)\right)^{1/2} / \left(1 + \left(\epsilon_r(0)\right)^{1/2}\right)\right) \left(1 + (i/6)\; 2 \left(\epsilon_r(0)\right)^{1/2} / \left(1 - \epsilon_r(0)\right) / w_0 \right.$$

$$\left. - \,(1/w_0^2) \cdot \left((0.083)2 \cdot \left(\epsilon_r(0)\right)^{1/2} / \left(1 - \epsilon_r(0)\right) + (0.028)2 \cdot \epsilon_r(0) / \left(1 - \epsilon_r^2(0)\right)\right) \ldots\right) . \quad (43)$$

The first term in the brackets clearly corresponds to the WKB approximation. It is simply the reflection coefficient for a homogeneous semi infinite medium with relative dielectric constant

$$\epsilon_r(0) = \epsilon'(0)/\epsilon_0 - i\,\epsilon''(0)/\epsilon_0 ,$$

and as shown above there are no contributions to the scattering reflections from the inhomogeneous medium in the WKB approximation. One can also prove the absence of scattering reflections by solving the boundary value problem of the semi infinite inhomogeneous medium in the WKB approximation*. The result, as anticipated is simply the reflection coefficient for a homogeneous medium of dielectric constant $\epsilon_r(0)$. The second and third terms of Eq. (43) are complex constants whose magnitude and argument represent the amplitude and phase, respectively, of the scattering reflections emerging at the interface and propagating in the negative z-direction. In other words the first secondary reflection due to the primary progressive wave, (see Eq. (42)),

$$E_y = C_1 \left(k(z)\right)^{-1/2} \exp\left(-i \int_0^z k(z)\, dz\right) \quad \dagger \tag{44}$$

in the inhomogeneous medium is given by,

$$\frac{1-\left(\epsilon_r(0)\right)^{1/2}}{1+\left(\epsilon_r(0)\right)^{1/2}} \cdot \frac{2 \cdot \left(\epsilon_r(0)\right)^{1/2}}{\left(1-\epsilon_r(0)\right)} \cdot \frac{i}{6\, w_0} \cdot \exp(i\, k_0\, z). \tag{45}$$

These secondary reflections combine destructively or constructively depending on the phase relative to the primary reflection at the interface due to the WKB term. See Eq. (43).

We have thus demonstrated the presence of secondary plane wave reflections in the homogeneous subspace $-\infty \leq z \leq 0$ and by inference secondary reflections in the inhomogeneous subspace $0 \leq z < \infty$.

V. THE RADIATION FIELD IN THE SEMI INFINITE INHOMOGENEOUS MEDIUM

In the WKB approximation (see Eq. (42)) and the Bremmer solution for an infinite inhomogeneous medium waves in the positive and negative direction can

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*See Appendix.

† It may be shown that the exact solution for E_y (see Eq. (10)) in terms of $H_{1/3}^{(2)}(w)$ reduces to a first approximation to the WKB solution, Eq. (44). See Reference 18.

still be identified, although some may be independent of each other as shown in the preceding section. The exact solution of the wave equation for the specific case of a linearly increasing dielectric constant in a semi infinite space as given by Eq. (10),

$$E_y = B w^{1/3} H_{1/3}^{(2)}(w), \qquad 0 \le z < \infty,$$

no longer permits a clear cut distinction between reflected and incident waves, or even waves travelling in positive and negative directions independently of each other.

In order to get a more physical picture of the wave field in the semi infinite space, and in addition the powerflow into the medium the complex Poynting vector S^* was machine calculated from

$$|\mathrm{Re}\ \vec{S}_z^*| = |\mathrm{Re}\ \frac{1}{2}(\vec{E} \times \tilde{\vec{H}})| = \mathrm{Re}\ \frac{1}{2} E_y \tilde{H}_x. ^{19} \tag{46}$$

From Eq. (10) and from Maxwell's equation for $H_x = (i\omega\mu_0)^{-1}\, \partial E_y / \partial z$, we find for normal incidence, $\theta_0 = 0$,

$$H_x = (i\omega\mu_0)^{-1} B w^{1/3} H_{-2/3}^{(2)}(w)\, k_0 (\xi^{1/2}). \tag{47}$$

Then,

$$\mathrm{Re}\ S_z^* = \mathrm{Re}\ \frac{1}{2}(-i\omega\mu_0)^{-1} B\tilde{B}\ |w^{1/3}|^2 H_{1/3}^{(2)}(w)\, \widetilde{H_{-2/3}^{(2)}}(w)\, k_0\, \widetilde{\xi^{1/2}}, \tag{48}$$

where the sign $\sim$ denotes the complex conjugate.

If the medium is not lossy machine calculations show that Re S_z^* is a constant for all $z > 0^{\dagger}$. The discussion of internal reflections throughout the inhomogeneous medium of the preceding section would have led one to guess at a constantly

† On the basis of a number of computer runs for very large a's (see Eq. (5)) (the gradient of the dielectric constant is proportional to a), and distances up to 5.0 cm from the interface.

decreasing Re S_z^* as $z \to \infty$. But some reflection shows that in fact a constant Poynting vector must be correct. If Re $S_z^* \to 0$ as $z \to \infty$, and no energy is dissipated in the medium all the reflected energy in the inhomogeneous medium would eventually have to emerge at the interface with the homogeneous medium, resulting in a reflection coefficient of 1. This is absurd. And contrary to intuition the energy flux in the positive z-direction stays constant as in a homogeneous lossless dielectric medium.†

Results of calculations of Re S_z^* when the medium is lossy, $\epsilon''(z) \neq 0$, are shown in Figure 7 for three different dielectric constant gradients. Curves I and II of Figure 7 correspond roughly to an Arizona loamy soil with approximately 15% and 2% moisture by weight at the interface. It is clear from the figure that the functional dependance of the powerflow on penetration distance is not the same for all curves. Curve I happens to be very closely exponential, Curve III roughly follows an exp $(-ax^2)$ decay.

Losses in the inhomogeneous medium may be calculated directly from the expression[21],

$$\text{Loss} = \frac{1}{2} \int_0^z E_y \tilde{E}_y \sigma \, dz, \tag{49}$$

where $\sigma(z)$ is the conductivity of the medium, which may be expressed in terms of the dielectric constant $\sigma(z) = \epsilon''(z)\,\omega$.[22] Using Eq. (10) for E_y, and the expression for $\sigma(z)$,

$$\text{Loss} = \frac{1}{2} \omega \epsilon''(0) \int_0^z (1 + az) \, |B|^2 \cdot |w^{1/3}|^2 \cdot |H_{1/3}^{(2)}(w)|^2 \, dz. \tag{50}$$

This integral was machine calculated by Simpson's rule. The calculations show the normalized L to be exactly equal to (1 - Re S_z^*). Additional proof that any decrease in the Poynting flux Re S_z^* in the inhomogeneous medium is solely due to the conduction (heating) losses and not any scattering reflection losses.

† This result was anticipated by Wallot, who gave an analytic proof of the constancy of the energy flux in the transition region between two homogeneous media for a real dielectric constant. See reference 20.

VI. CONCLUSIONS

We have shown that the reflection coefficient of a semi infinite inhomogeneous medium with constant positive gradient $d\epsilon(z)/dz = \epsilon'(0)\,a - i\,\epsilon''(0)\,b$ is not necessarily a monotonically increasing function for increasing a and b. Machine calculations of the reflection coefficient $|R|$, show the coefficient to decrease with increasing b until a minimum is reached. On the other hand $|R|$ seems to increase monotonically with a for a given b as far as can be judged from the available machine calculations.

We demonstrated the presence of scattering reflections in the free space $-\infty < z \leq 0$, emerging from the inhomogeneous halfspace, in analogy with Bremmer's results on infinitely and semi infinitely extensive media. But in addition we drew the important conclusion that these scattering reflections, which are coherent, interfere to produce the calculated drop in the reflection coefficient despite increasing gradients of the dielectric constant. As a consequence we find the startling result that a semi infinite inhomogeneous medium with a steep dielectric constant gradient may have a much smaller reflection coefficient than the same dielectric half space with zero gradient, i.e. $a = b = 0$, and $\epsilon(z) = \epsilon'(0) - i\,\epsilon''(0)$.

APPENDIX

We shall consider the reflection of a plane horizontally polarized wave, at normal incidence, from an inhomogeneous semi infinite medium in the WKB approximation. The dielectric properties of the inhomogeneous medium are characterized by the propagation constant $k^2(z) = \omega^2 \mu_0 \epsilon(z)$. (See Eq. (21).) Then from Eqs. (42) and (44), we have for the progressive wave in the inhomogeneous medium,

$$E_y = C\left(k(z)\right)^{-1/2} \exp\left(-i\int_0^z k(z)\,dz\right), \tag{A1}$$

and in the homogeneous medium as before from Eq. (3),

$$E_y = \exp(-i k_0 z) + R \exp(+ i k_0 z). \tag{A2}$$

From the continuity conditions at $z = 0$, we find from Eqs. (A1) and (A2),

$$1 + R = C/\left(k(0)\right)^{1/2} \tag{A3}$$

$$-1 + R = \left.\frac{C\,dk(z)/dz}{i k_0 \left(k(0)\right)^{3/2}} - \frac{C\,k(0)}{\left(k(0)\right)^{1/2} k_0}\right|_{z=0} \tag{A4}$$

In the WKB approximation $\left(dk(z)/dz\right)/k(z)^{3/2} \sim 0$, then solving Eqs. (A3), and (A4) for R, we find,

$$R = \frac{k_0 - \left(k(0)\right)}{k_0 + \left(k(0)\right)} . \tag{A5}$$

Equation (A5) is the reflection coefficient for a homogeneous medium with dielectric constant $\epsilon(0)$. Consequently, in the WKB approximation there are no scattering reflections in the inhomogeneous medium.

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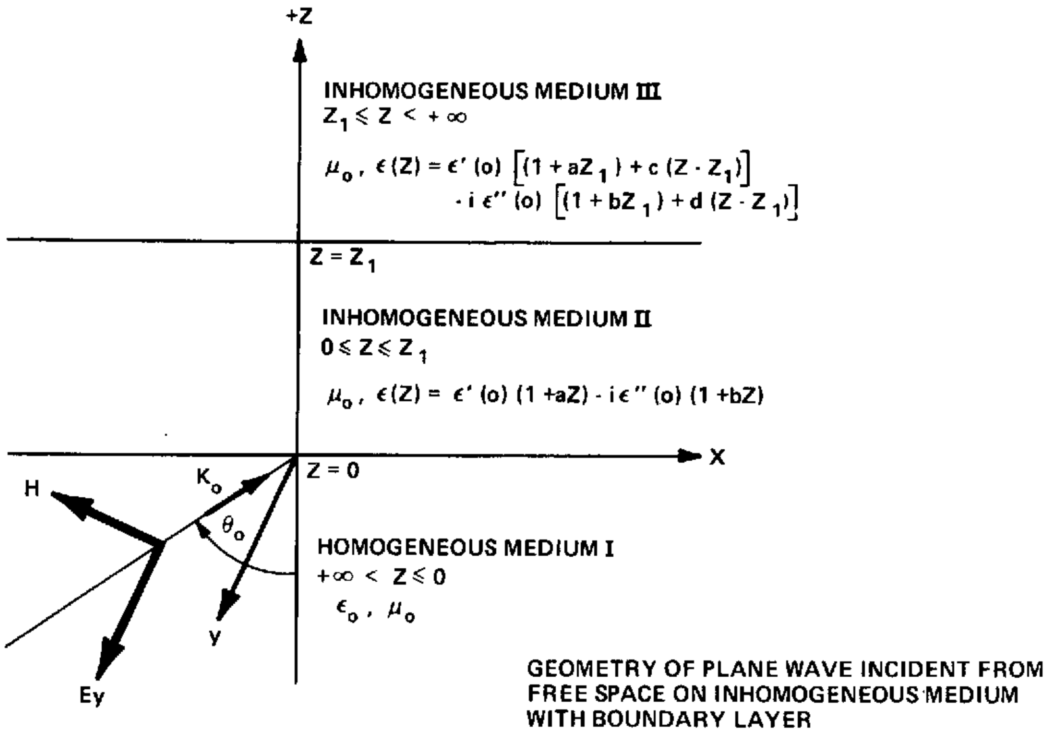

Figure 1. Geometry of a plane wave incident from free space on the x-y face of a semi infinite inhomogeneous medium. The plane of incidence is the x-z plane and the wave is polarized in the y-direction. There is an inhomogeneous transition layer between $0 \leq z \leq z_1$ and both inhomogeneous media have a dielectric constant which is a linear complex function of z. Free space, the inhomogeneous layer and the inhomogeneous half space are designated as media I, II, and III respectively. The angle of incidence, θ_0, between the wave vector $\vec{k}_0$ and the normal to the plane is taken to be zero in all cases: The permiability and dielectric constants of free space are respectively μ_0 and ϵ_0.

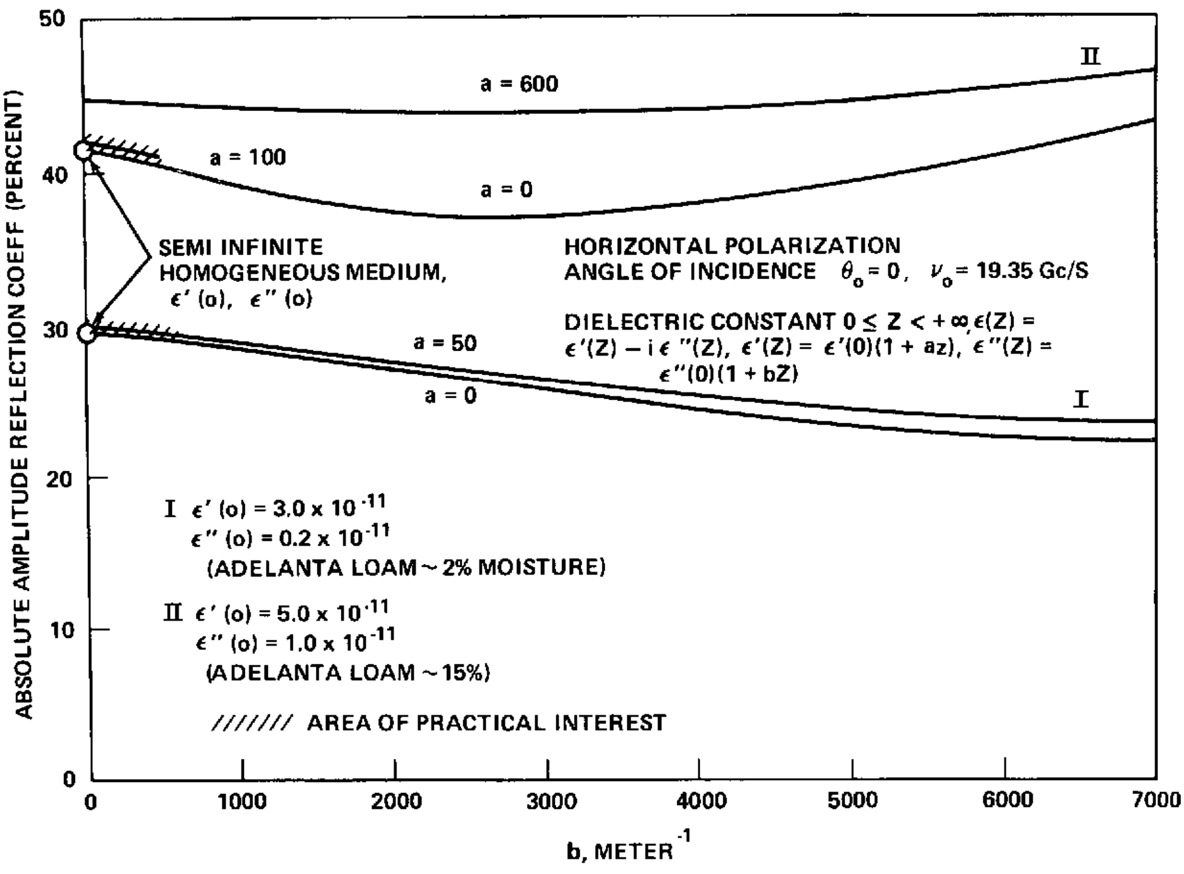

Figure 2. Absolute amplitude reflection coefficient for a semi infinite inhomogeneous medium with dielectric constant linearly dependent on z. No boundary layer; horizontal polarization; angle of incidence, 0°; frequency, 19.35 Gc/sec.

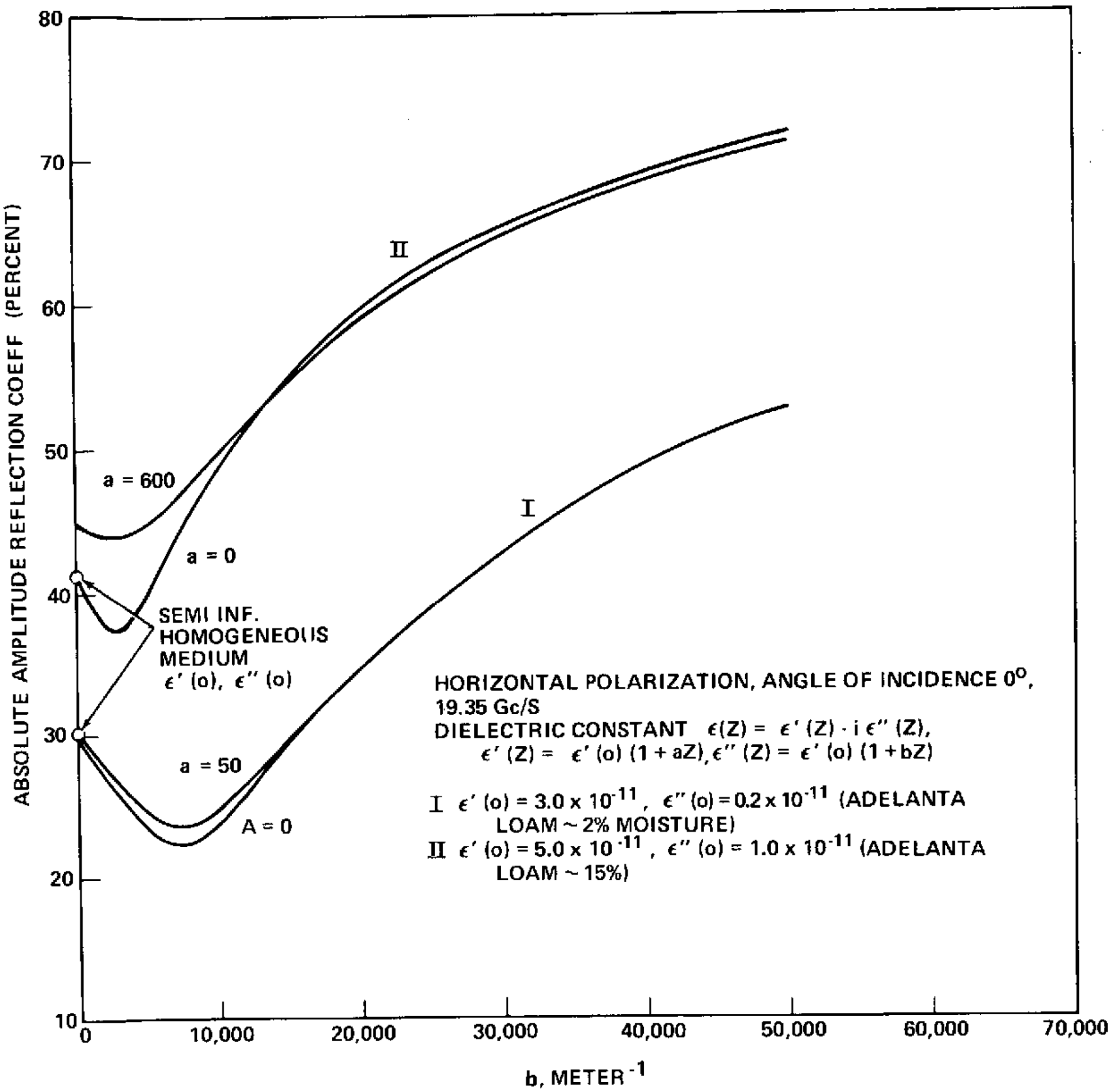

Figure 3. Absolute amplitude reflection coefficient for a semi infinite inhomogeneous medium with dielectric constant linearly dependent on z. No boundary layer; horizontal polarization; angle of incidence, 0°; frequency, 19.35 Gc/sec.

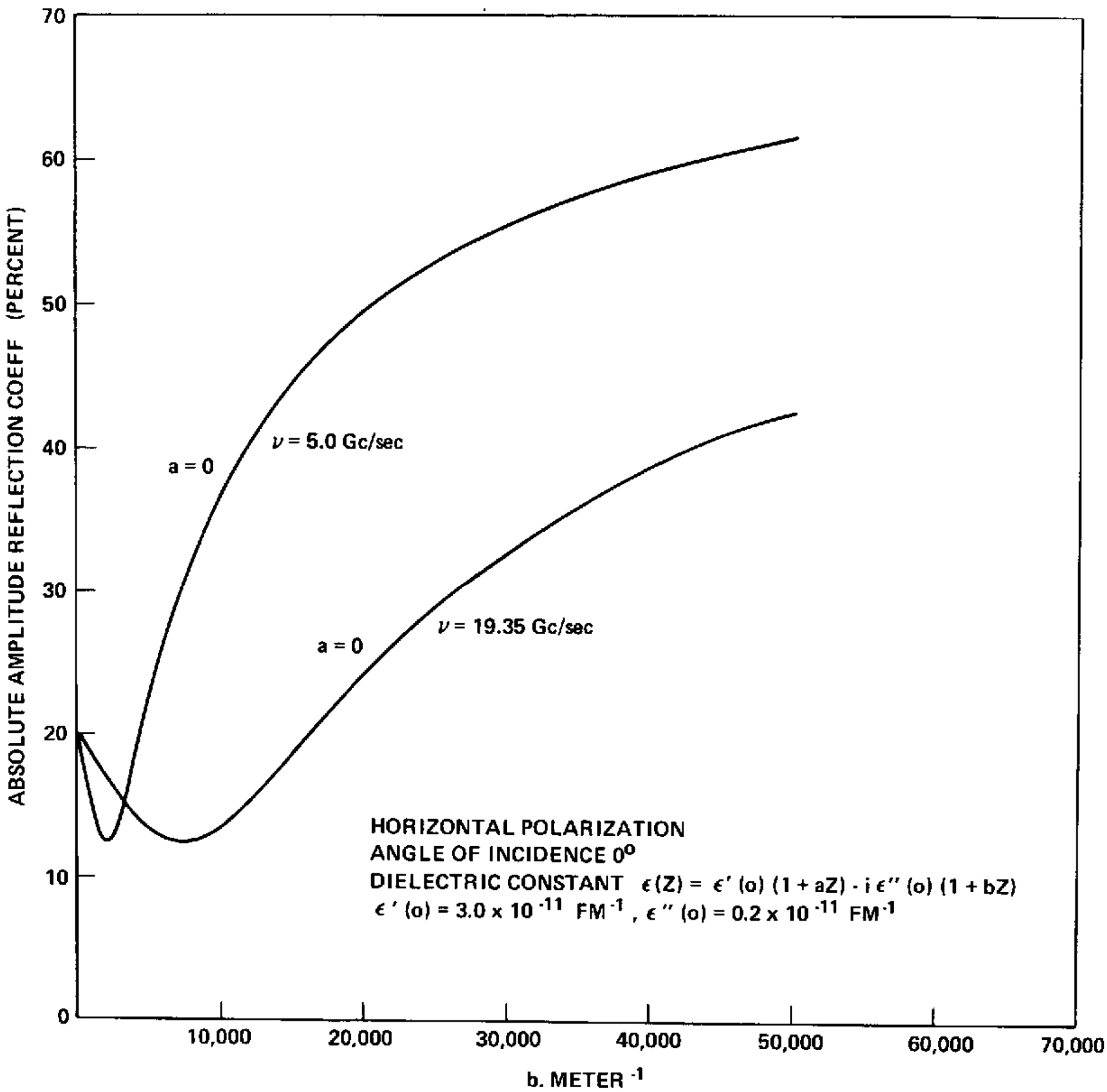

Figure 4. Absolute amplitude reflection coefficient for a semi infinite inhomogeneous medium with dielectric constant linearly dependent on z. No boundary layer; horizontal polarization; angle of incidence, 0°; frequency, 19.35, and 5.0 Gc/sec.

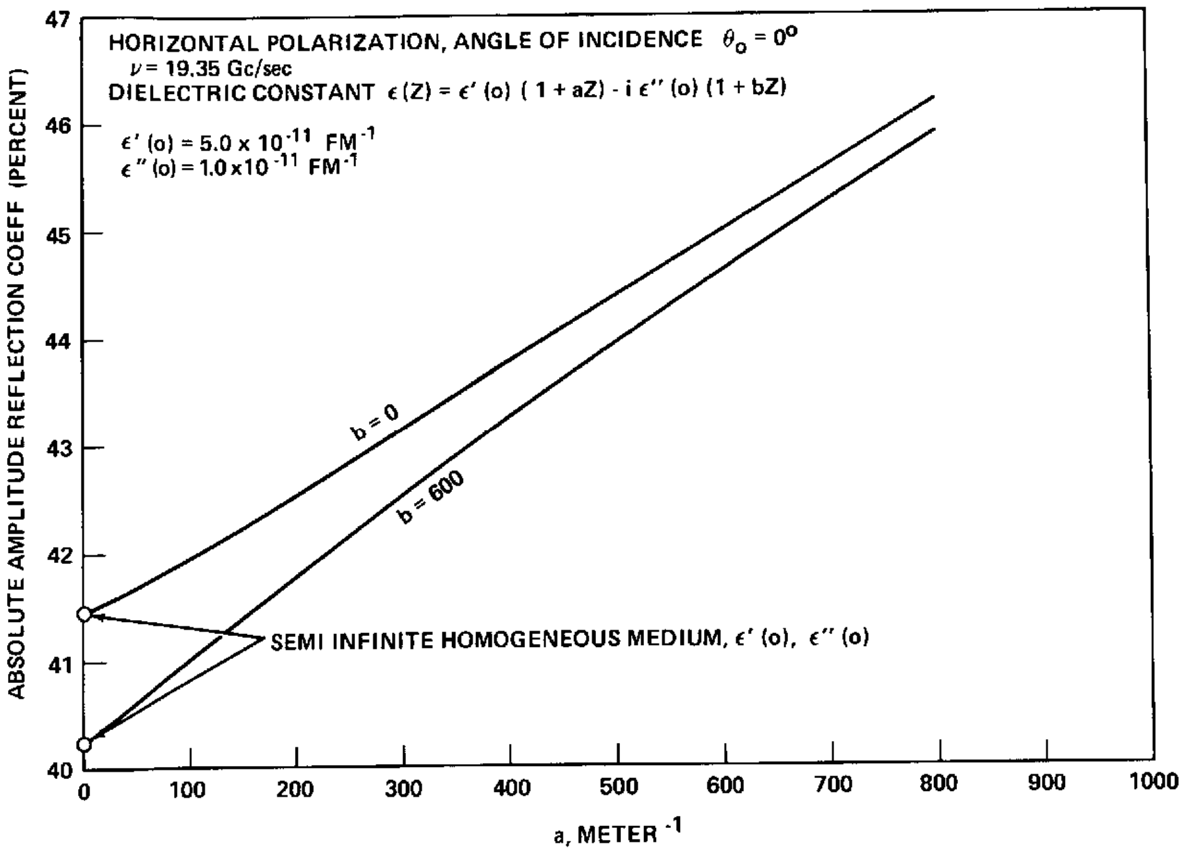

Figure 5. Absolute amplitude reflection coefficient for a semi infinite inhomogeneous medium, with dielectric constant linearly dependent on z, as a function of the parameter a. No boundary layer; horizontal polarization, angle of incidence, 0°; frequency, 19.35 Gc/sec.

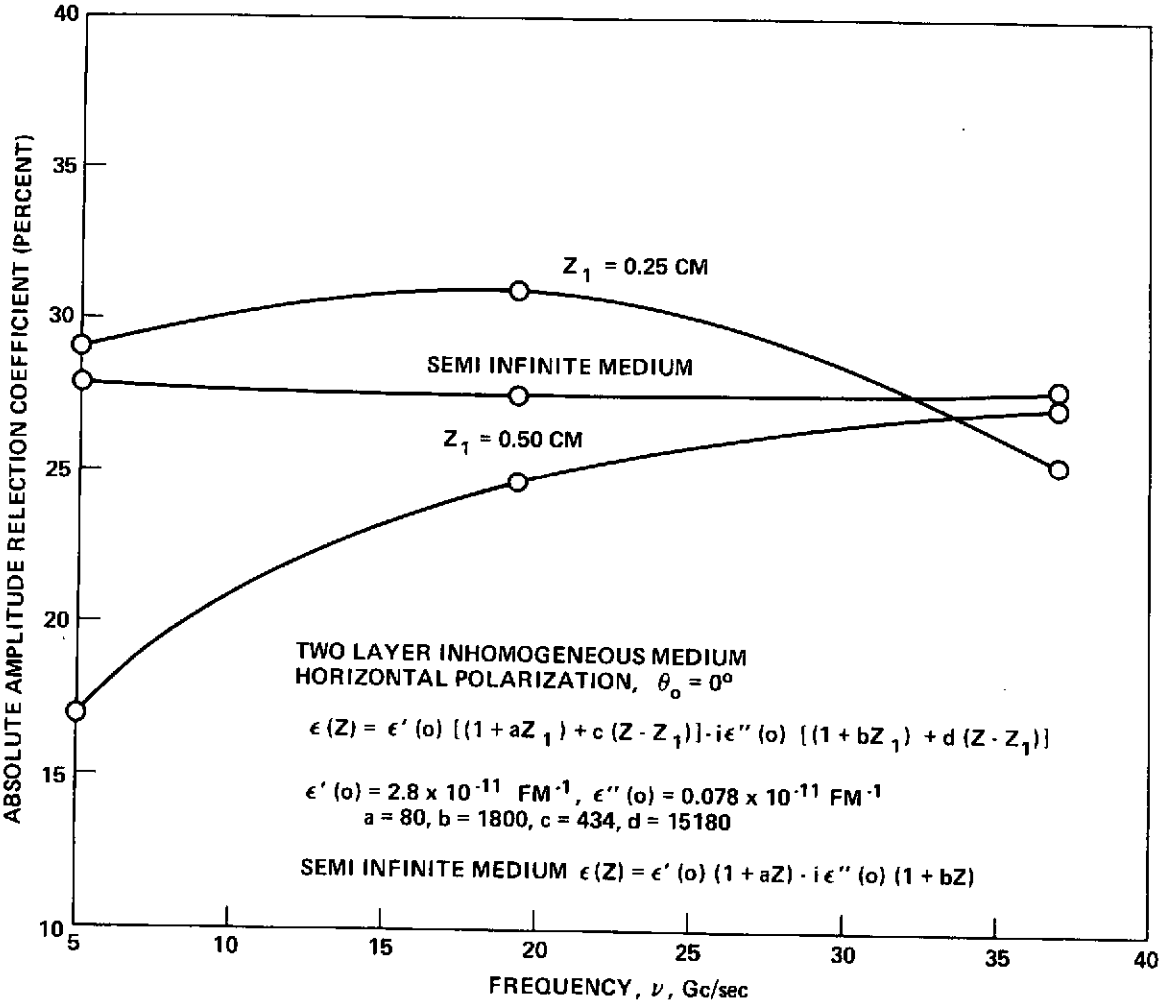

Figure 6. Absolute amplitude reflection coefficient for a two layer inhomogeneous medium as a function of frequency. Dielectric constant of inhomogeneous media linearly dependent on z. Horizontal polarization; angle of incidence, 0°.

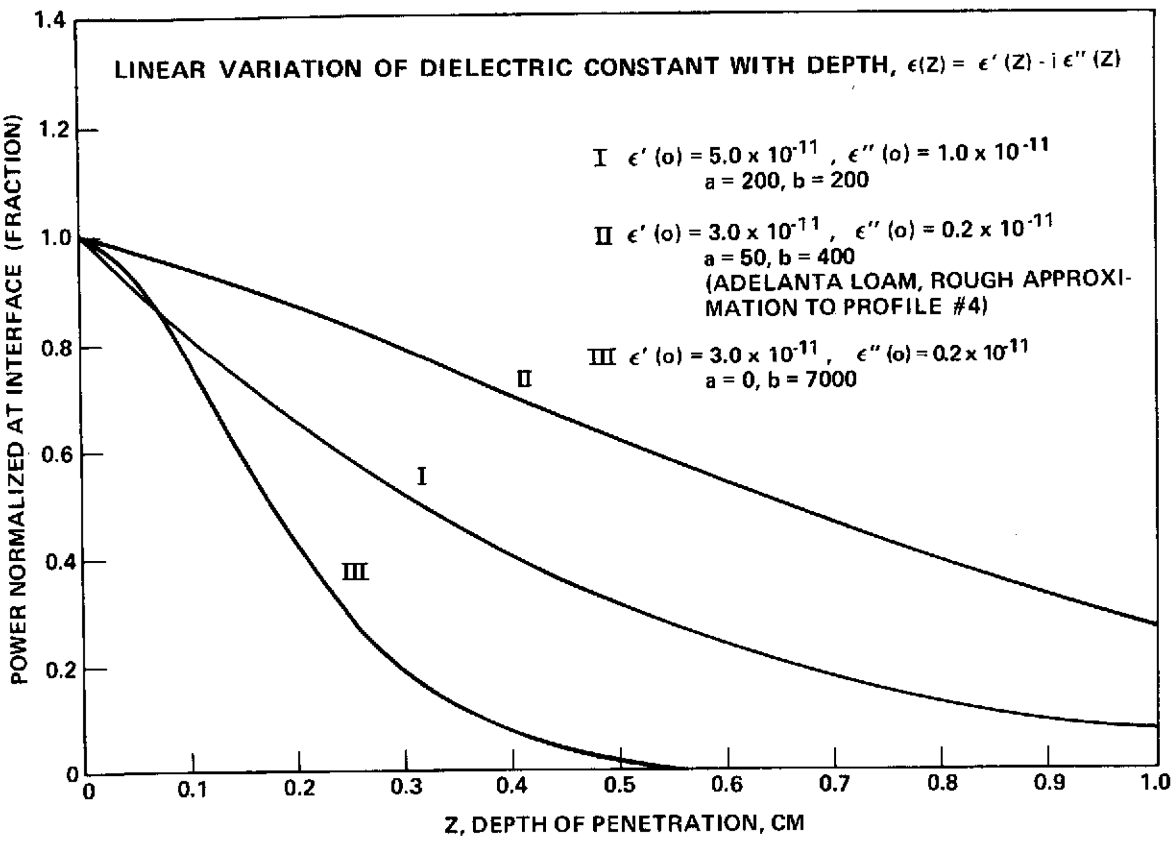

Figure 7. Power flow in semi infinite inhomogeneous medium with no boundary layer as a function of the depth of penetration. Frequency, 19.35 Gc/sec, normal incidence.